ON THE HILBERT SERIES OF VERTEX COVER ALGEBRAS OF COHEN-MACAULAY BIPARTITE GRAPHS

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Abstract. We study the Hilbert function and the Hilbert series of the vertex cover algebra $A(G)$, where G is a Cohen-Macaulay bipartite graph.

MSC: 05E40, 13P10.

Keywords: Cohen-Macaulay bipartite graph, Vertex cover, Hilbert series.

1. Introduction

Let $G = (V, E)$ be a simple (i.e., finite, undirected, loop less and without multiple edges) graph with the vertex set $V = [n]$ and the edge set $E = E(G)$. A *vertex cover* of G is a subset $C \subset V$ such that $C \cap \{i, j\} \neq \emptyset$, for any edge $\{i, j\} \in E(G)$. A vertex cover C of G is called *minimal* if no proper subset $C' \subset C$ is a vertex cover of G. A graph G is called *unmixed* if all minimal vertex covers of G have the same cardinality. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. The *edge ideal* of G is the monomial ideal $I(G)$ of R generated by all the quadratic monomials $x_i x_j$ with $\{i, j\} \in E(G)$. It is said that a graph G is $Cohen-Macaulay$ (over K) if the quotient ring $R/I(G)$ is Cohen-Macaulay. Every Cohen-Macaulay graph is unmixed.

A vertex cover $C \subset [n]$ can be represented as a $(0, 1)-$vector c that satisfies the restriction $c(i) + c(j) \geq 1$, for every $\{i, j\} \in E(G)$. For each $k \in \mathbf{N}$, a *vertex cover* of G of *order* k, or simply a $k-$*vertex cover* of G, is a vector $c \in \mathbf{N}^n$ such that $c(i) + c(j) \geq k$, for every $\{i, j\} \in E(G)$. The *vertex cover algebra* $A(G)$ is defined as the subalgebra of the one variable polynomial ring $R[t]$ generated by all monomials $x_1^{c_1} \cdots x_n^{c_n} t^k$, where $c = (c_1, \ldots, c_n) \in \mathbf{N}^n$ is a k-vertex cover of G. This algebra was introduced and first studied in [5]. Let $\mathfrak{m}$ be the maximal graded ideal of R. The graded K-algebra $\bar{A}(G) = A(G)/\mathfrak{m}A(G)$ is called the *basic cover algebra* and it was introduced and first studied in [4, Section 3].

Our aim in this paper is to study the Hilbert function and series of the vertex cover algebra $A(G)$ for Cohen-Macaulay bipartite graphs.

Let $P_n = \{p_1, p_2, \ldots, p_n\}$ be a poset with a partial order $\leq$. Let $G = G(P_n)$ be the bipartite graph on the set $V_n = W \cup W'$, where $W = \{x_1, x_2, \ldots, x_n\}$ and $W' = \{y_1, y_2, \ldots, y_n\}$, whose edge set $E(G)$ consists of all 2-element subsets $\{x_i, y_j\}$ with $p_i \leq p_j$. It is said that a bipartite graph G on $V_n = W \cup W'$ *comes from a poset*, if there exists a finite poset P_n on $\{p_1, p_2, \ldots, p_n\}$ such that $p_i \leq p_j$ implies $i \leq j$, and after relabeling of the vertices of G one has $G = G(P_n)$. Herzog and Hibi

proved in [3] that a bipartite graph G is Cohen-Macaulay if and only if G comes from a poset.

In Section 2, we firstly notice that the Hilbert function and series of the vertex cover algebras $A(G)$ are invariant to poset isomorphisms. We obtain a recurrence relation for the minimal vertex covers of a Cohen-Macaulay graph G and we study the Hilbert function of $A(G)$.

In Section 3, we study the Hilbert series of $A(G)$. For a poset $P_n = \{p_1, p_2, ..., p_n\}$ we denote by $\mathcal{J}(P_n)$ the lattice of all poset ideals of P_n. For each subset $\emptyset \neq F \subset [n]$ we denote by $P_n(F)$ the subposet of P_n induced by the subset $\{p_i | i \in F\}$ and by G_F the bipartite graph that comes from $P_n(F)$. The main result of this paper is given in Theorem 3.4, which shows that one may reduce the computation of the Hilbert series of the vertex cover algebra $A(G)$ to the computation of the Hilbert series of the basic cover algebra $\bar{A}(G_F)$, for all $F \subset [n]$. If $F = \emptyset$, then, by convention, the Hilbert series of $\bar{A}(G_F)$ is equal to $\frac{1}{1-z}$. Namely, we have the following formula:

$$H_{A(G)}(z) = \frac{1}{(1-z)^n} \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z}\right)^{n-|F|}.$$

Moreover, we give a combinatorial interpretation for the h-vector of $A(G)$ in terms of the poset P_n. Using this interpretation we show that the h-vector of $A(G)$ is unimodal. We give bounds for its components and derive bounds for $e(A(G))$, the multiplicity of $A(G)$.

We show that both chains and antichains are uniquely determined up to a poset isomorphism by the Hilbert series of their corresponding vertex cover algebras.

2. Vertex cover algebras of Cohen-Macaulay bipartite graphs

Let $S = K[x_1, ..., x_n, y_1, ..., y_n]$ and let $G = G(P_n)$, where $P_n = \{p_1, ..., p_n\}$ is a poset with a partial order $\leq$. We recall that, by [5], the vertex cover algebra $A(G)$ is standard graded over S and it is the Rees algebra of the *cover ideal* I_G, which is generated by all monomials $x_1^{c_1} \cdots x_n^{c_n} y_1^{c_{n+1}} ... y_n^{c_{2n}}$, where $c = (c_1, \ldots, c_{2n})$ is a 1-vertex cover of G. Thus

$$A(G) = S \oplus I_G t \oplus \ldots \oplus I_G^k t^k \oplus \ldots$$

Let $\{m_1, m_2, ..., m_l\}$ be the minimal system of generators of I_G. We view $A(G)$ as a standard graded K-algebra by assigning to each x_i and y_j, $1 \leq i, j \leq n$ and to each $m_k t$, $1 \leq k \leq l$, the degree 1. Since each monomial m_k corresponds to a minimal vertex cover of G of cardinality n, the Hilbert function of $A(G)$ is given by

$$H(A(G), k) = \sum_{j=0}^{k} \dim_K (I_G^j)_{jn+(k-j)}, \text{ for all } k \geq 0. \tag{1}$$

Remark 2.1. Let $P_n = \{p_1, ..., p_n\}$ and $P'_n = \{p'_1, ..., p'_n\}$ be two isomorphic finite posets and let $G = G(P_n)$ and $G' = G(P'_n)$. Then the cover ideals I_G and $I_{G'}$ are isomorphic as graded K-vector spaces and, consequently, the Hilbert function and series of $A(G)$ and $A(G')$ coincide. Let $f : P_n \to P'_n$ be a poset isomorphism (i.e., f is a bijective map with $p_i \leq p_j$ if and only if $f(p_i) \leq f(p_j)$). Then f induces a

permutation g of $[n]$, $i \to g(i)$, defined by $p'_{g(i)} = f(p_i)$, for every $i \in [n]$. We notice that

$$p_i \le p_j \Leftrightarrow f(p_i) \le f(p_j) \Leftrightarrow p'_{g(i)} \le p'_{g(j)}, \tag{2}$$

and we define a map $h : V(G) \to V(G')$ as follows:

$$\begin{aligned} h(x_i) &= x_{g(i)}, \text{ if } i \in [n], \\ h(y_j) &= y_{g(j)}, \text{ if } j \in [n]. \end{aligned}$$

Then h induces a K-automorphism of S which maps I_G onto $I_{G'}$, hence, I_G and $I_{G'}$ are isomorphic as graded K-vector spaces. By (1), we also have

$$H(A(G'), k) = \sum_{j=0}^{k} \dim_K (I^j_{G'})_{jn+(k-j)}, \text{ for all } k \ge 0.$$

Since the powers I^j_G and $I^j_{G'}$ are isomorphic as graded K-vector spaces as well, for all $j \ge 1$, we get $H(A(G), k) = H(A(G'), k)$, for all $k \ge 0$.

We denote by $\mathcal{M}(G)$ the set of minimal vertex covers of a graph G. Vertex covers and stable sets of a graph G are dual concepts, that is, a subset $C \subset V(G)$ is a vertex cover of G if and only if the complement set $V(G) \backslash C$ is a stable set of G ([7]). Next, inspired by [7, Lemma 2.5], we give a recurrence relation to obtain the set of the minimal vertex covers of a Cohen-Macaulay bipartite graph G_n which comes from a poset $P_n = \{p_1, \ldots, p_n\}$. We denote by G_{n-1} the subgraph of G_n which comes from the poset $P_{n-1} = \{p_1, \ldots, p_{n-1}\}$ and by V_{n-1} the set $\{x_1, \ldots, x_{n-1}\} \cup \{y_1, \ldots, y_{n-1}\}$.

Proposition 2.2. *Let $G_n = G(P_n)$, where $P_n = \{p_1, \ldots, p_n\}$, $n \ge 2$, is a poset such that $p_i \le p_j$ implies $i \le j$. Then a subset $C_n \subset V_n$ is a minimal vertex cover of G_n if and only if either $C_n = C_{n-1} \cup \{y_n\}$, where $C_{n-1} \subset V_{n-1}$ is a minimal vertex cover of G_{n-1} or $C_n = C_{n-1} \cup \{x_n\}$, where $C_{n-1} \subset V_{n-1}$ is a minimal vertex cover of G_{n-1} such that $x_i \in C_{n-1}$ for each $i \in [n-1]$ with $p_i \le p_n$.*

Proof. 'If' it is straightforward.

Let us proof 'Only if'. Since G_n is a Cohen-Macaulay graph, it is unmixed and all its minimal vertex covers have the same cardinality, namely n, for every $n \ge 2$.

If $n = 2$ the statement obviously holds.

We assume that $n \ge 3$. Let $C_n = \{c_1, \ldots, c_n\}$ be a minimal vertex cover of G_n. Put $C_n = \{c_1, \ldots, c_n\}$, $C_{n-1} = C_n \cap V_{n-1}$ and $C'_n = C_n \cap \{x_n, y_n\}$. Obviously, $|C'_n| \le 2$.

If $|C'_n| = 0$, then $C_n \cap \{x_n, y_n\} = \emptyset$, which is impossible. Now let us suppose that $|C'_n| = 2$, hence $C'_n = \{x_n, y_n\}$ and $|C_{n-1}| = n-2$. Since C_n is a vertex cover of G_n, it follows that the intersection of C_n with every edge $\{x_i, y_j\}$ of the subgraph G_{n-1} $(1 \le i \le j \le n-1)$ is a nonempty subset of C_{n-1}, hence C_{n-1} is a vertex cover of G_{n-1} of cardinality $n-2$. But this is impossible since all minimal vertex covers of G_{n-1} have the cardinality equal to $n-1$.

It follows that $|C'_n| = 1$, $|C_{n-1}| = n-1$ and exactly one of the vertices x_n or y_n belongs to C_n. We can put, without loss of generality, either $c_n = x_n$ or $c_n = y_n$, and $C_{n-1} = \{c_1, \ldots, c_{n-1}\} \subset V_{n-1}$. Since C_n is a vertex cover of G_{n-1}, the intersection of

C_n with every edge $\{x_i, y_j\}$ of the subgraph G_{n-1} ($1 \leq i \leq j \leq n-1$) is a nonempty subset of C_{n-1}, hence C_{n-1} is a vertex cover of G_{n-1}. Moreover, C_{n-1} is a minimal vertex cover of G_{n-1}, since $|C_{n-1}| = n-1$.

If we choose $c_n = x_n$, then $y_n \notin C_n$. Since C_n is a vertex cover of G_n, it follows that $C_n \cap \{x_i, y_n\} = \{x_i\}$, for every $\{x_i, y_n\} \in E(G_n)$ with $i \in [n-1]$, which implies that $x_i \in C_n$, for each $i \in [n-1]$ with $\{x_i, y_n\} \in E(G_n)$. Hence $x_i \in C_n \cap V_{n-1} = C_{n-1}$, for each $i \in [n-1]$ with $p_i \leq p_n$.

If we choose $c_n = y_n$, then there is no (other) restriction on the minimal vertex cover C_{n-1} of G_{n-1}. $\square$

Remark 2.3. *Let G be a Cohen-Macaulay bipartite graph which comes from the poset P_n.* By [4, Theorem 2.1] there is a one-to-one correspondence between the set $\mathcal{M}(G)$ and the distributive lattice $\mathcal{J}(P_n)$ of all poset ideals of P_n. Thus it can be assigned to each minimal vertex cover C of G the poset ideal α_C of P_n that is defined as $\alpha_C = \{p_i | x_i \in C\}$. Conversely, if α is a poset ideal of P_n, then the corresponding set $C_\alpha = \{x_i | p_i \in \alpha\} \cup \{y_j | p_j \notin \alpha\}$ is a minimal vertex cover of G. By Proposition 2.2, one may give a recursive procedure to compute the lattice $\mathcal{J}(P_n)$.

For $C \in \mathcal{M}(G)$ we denote $m_C = (\prod\limits_{x_i \in C} x_i) \cdot (\prod\limits_{y_j \in C} y_j)$. If G is unmixed, then each $C \in \mathcal{M}(G)$ has exactly n vertices, hence, deg $m_C = n$, for all $C \in \mathcal{M}(G)$. The next result shows a property of monotony of the Hilbert function of an unmixed bipartite graph.

Proposition 2.4. *Let G, G' and G'' be unmixed bipartite graphs on V_n, $n \geq 1$, such that $E(G'') \subset E(G) \subset E(G')$. Then the following inequalities hold:*

$$H(A(G'), k) \leq H(A(G), k) \leq H(A(G''), k), \text{ for all } k \geq 0.$$

Proof. It is known ([5, Theorem 5.1.b]) that $I_G = (m_C \mid C \in \mathcal{M}(G))$. Similarly, we have $I_{G'} = (m_C \mid C \in \mathcal{M}(G'))$ and $I_{G''} = (m_C \mid C \in \mathcal{M}(G''))$. It follows that all the cover ideals are generated in the same degree n.

From the inclusions between the edge sets and the hypothesis of unmixedness, we get $\mathcal{M}(G') \subset \mathcal{M}(G) \subset \mathcal{M}(G'')$. Therefore, $I_{G'} \subset I_G \subset I_{G''}$. We also have

$$(I^a_{G'})_b \subset (I^a_G)_b \subset (I^a_{G''})_b, \tag{3}$$

for all integers $a \geq 1$ and $b \geq 0$, which, by (1), implies the desired inequalities. $\square$

It is obvious that, for the Cohen-Macaulay bipartite graphs, the chain provides the largest number of edges and the antichain the smallest number of edges.

Corollary 2.5. *Let G be a Cohen-Macaulay bipartite graph on V_n, $n \geq 1$. Then the following inequalities hold:*

$$H(A(G'), k) \leq H(A(G), k) \leq H(A(G''), k), \text{ for all } k \geq 0, \tag{4}$$

where G' and G'' are bipartite graphs on V_n that come from a chain, respectively, an antichain with n elements.

Proof. Let G, G', respectively, G'' be graphs that come from a poset $P_n = \{p_1, .., p_n\}$, a chain $P'_n = \{p'_1, .., p'_n\}$, respectively, an antichain $P''_n = \{p''_1, .., p''_n\}$. By Remark 2.1

we may asume that $p_i \leq p_j$ and $p'_i \leq p'_j$ imply $i \leq j$. It is straightforward to notice that $E(G'') \subset E(G) \subset E(G')$. Therefore, by applying Proposition 2.4, the desired inequalities follow. $\square$

The next result stresses a property of monotony for the multiplicity of the vertex cover algebra for unmixed bipartite graphs.

Corollary 2.6. *Let G, $G'$and G'' be unmixed bipartite graphs on V_n such that $E(G'') \subset E(G) \subset E(G')$. Then the following inequalities hold:*

$$e(A(G')) \leq e(A(G)) \leq e(A(G'')).$$

Proof. By Proposition 2.4 we have $H(A(G')), k) \leq H(A(G), k) \leq H(A(G''), k)$, for all $k \geq 0$. Since $H(A(G), k)$, $H(A(G'), k)$, respectively, $H(A(G''), k)$ are all polynomials of degree $2n$ (since $\dim A(G) = \dim S + 1 = 2n + 1$ [2]) with the leading coefficients $\frac{e(A(G))}{(2n)!}$, $\frac{e(A(G'))}{(2n)!}$, respectively, $\frac{e(A(G''))}{(2n)!}$, the conclusion follows. $\square$

3. The Hilbert series of vertex cover algebras of Cohen-Macaulay bipartite graphs

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables over a field K and let $G = G(P_n)$, where $P_n = \{p_1, ..., p_n\}$ is a poset such that $p_i \leq p_j$ implies $i \leq j$.

We denote $B_G = K[\{x_i\}_{1\leq i\leq n}, \{y_j\}_{1\leq j\leq n}, \{u_\alpha\}_{\alpha\in\mathcal{J}(P_n)}]$. The *toric ideal* Q_G of $A(G)$ is the kernel of the surjective homomorphism $\varphi : B_G \to A(G)$ defined by $\varphi(x_i) = x_i$, $\varphi(y_j) = y_j$, $\varphi(u_\alpha) = m_\alpha t$, where $m_\alpha = (\prod\limits_{p_i\in\alpha} x_i) \cdot (\prod\limits_{p_j\notin\alpha} y_j)$, $\alpha \in \mathcal{J}(P_n)$, are the minimal monomial generators of the cover ideal I_G.

Let $<_{lex}$ denote the lexicographic order on $K[\{x_i\}_{1\leq i\leq n}, \{y_j\}_{1\leq j\leq n}]$ induced by the ordering $x_1 > .. > x_n > y_1 > .. > y_n$ and $<^\#$ the reverse lexicographic order on $K[\{u_\alpha\}_{\alpha\in\mathcal{J}(P_n)}]$ induced by an ordering of the variables u_α's such that $u_\alpha > u_\beta$ if $\beta \subset \alpha$ in $\mathcal{J}(P_n)$. Let $<^\#_{lex}$ be the monomial order on B_G defined as the product of the monomial orders $<_{lex}$ and $<^\#$ from above. The reduced Gröbner basis $\mathcal{G}$ of the toric ideal Q_G of $A(G)$ with respect to the monomial order $<^\sharp_{lex}$ on B_G was computed in [3, Theorem 1.1]:

$$\mathcal{G} = \{\underline{x_j u_\alpha} - y_j u_{\alpha\cup\{p_j\}}, j \in [n], \alpha \in \mathcal{J}(P_n), p_j \notin \alpha, \alpha \cup \{p_j\} \in \mathcal{J}(P_n),$$
$$\underline{u_\alpha u_\beta} - u_{\alpha\cup\beta}u_{\alpha\cap\beta}, \alpha, \beta \in \mathcal{J}(P_n), \alpha \not\subset \beta, \beta \not\subset \alpha\},$$

where the initial monomial of each binomial of $\mathcal{G}$ is the first monomial.

Let $S_G = K[\{u_\alpha\}_{\alpha\in\mathcal{J}(P_n)}]$ be the polynomial ring in $|\mathcal{J}(P_n)|$ variables over K, let $\bar{A}(G)$ the basic vertex cover algebra and $\Delta(\mathcal{J}(P_n))$ the order complex of the lattice $(\mathcal{J}(P_n), \subset)$ whose vertices are the chains of P_n. (We refer the reader to [1], [4, Section 3] for the definition and properties of the basic cover algebra associated to a graph and [2, §5.1] for the definition and properties of the order complex of a poset.) The *toric ideal* $\bar{Q}_G$ of $\bar{A}(G)$ is the kernel of the surjective homomorphism $\pi : S_G \to \bar{A}(G)$, $\pi(u_\alpha) = m_\alpha$. The reduced Gröbner basis $\mathcal{G}_0$ of $\bar{Q}_G$ with respect to $<^\#$ on S_G was computed in [4, Theorem 3.1]:

$$\mathcal{G}_0 = \{\underline{u_\alpha u_\beta} - u_{\alpha\cup\beta}u_{\alpha\cap\beta} | \alpha, \beta \in \mathcal{J}(P_n), \alpha \not\subset \beta, \beta \not\subset \alpha\},$$

where the initial monomial of each binomial of $\mathcal{G}_0$ is the first monomial.

Proposition 3.1. *The graded K-algebra $\bar{A}(G)$ and the the order complex $\Delta(\mathcal{J}(P_n))$ have the same h-vector.*

Proof. $\bar{Q}_G$ is a graded ideal (generated by binomials) and the initial ideal $\mathrm{in}_{<\#}(\bar{Q}_G)$ of the toric ideal $\bar{Q}_G$ coincides with the Stanley-Reisner ideal $I_{\Delta(\mathcal{J}(P_n))}$, hence $S_G/\bar{Q}_G$ and $K[\Delta(\mathcal{J}(P_n))]$ have the same h-vector. Since $S_G/\bar{Q}_G \simeq \bar{A}(G)$ as graded K-algebras, the conclusion follows. □

Remark 3.2. Since $\mathcal{J}(P_n)$ is a full sublattice of the Boolean lattice $\mathcal{L}_n$ on the set $\{p_1, p_2, \ldots, p_n\}$ ([4, Theorem 2.2.]), it follows that $\dim \Delta(\mathcal{J}(P_n)) = n$. Let $h = (h_0, h_1, ..., h_{n+1})$ be the h-vector of $\Delta(\mathcal{J}(P_n))$ and $\bar{A}(G)$. As we noticed above, the basic vertex cover algebra $\bar{A}(G)$ can be identified with the Hibi ring $S_G/\bar{Q}_G$, which arises from the distributive lattice $\mathcal{J}(P_n)$. The i-th component h_i of the h-vector of $S_G/\bar{Q}_G$ and, consequently, of $\bar{A}(G)$ is equal to the number of linear extensions of P_n, which, seen as permutations of $[n]$, have exactly i descents ([6]). In particular,

$$h_i \geq 0, \text{ for all } 0 \leq i \leq n-1, \; h_0 = 1, \text{ and } h_n = h_{n+1} = 0. \tag{5}$$

For example, if $P''_n = \{p''_1, ..., p''_n\}$ is an antichain, then each permutation of $[n]$ can be seen as a linear extension of P''_n, hence, for all $0 \leq i \leq n-1$, the i-th component of the h-vector of $\Delta(\mathcal{J}(P''_n))$ is equal to the number of all permutations of $[n]$ with exactly i descents, which is the Eulerian number $A(n,i)$.

For each $\emptyset \neq F \subset [n]$ we denote by $P_n(F)$ the subposet of P_n induced by the subset $\{p_i | i \in F\}$. The main result of the paper relates the Hilbert series of $A(G)$ to the Hilbert series of $\bar{A}(G_F)$, for all $F \subset [n]$, where G_F denotes the bipartite graph that comes from the poset $P_n(F)$. If $F = \emptyset$, then, by convention, the Hilbert series of $\bar{A}(G_F)$ is equal to $\frac{1}{1-z}$.

In order to prove the main theorem we need a preparatory result.

Let $\emptyset \neq F \subsetneq [n]$ and let α be a poset ideal of $P_n(\bar{F})$, where by $\bar{F}$ we mean the complement of F in $[n]$. We denote by δ_α the maximal subset of $P_n(F)$ such that $\alpha \cup \delta_\alpha \in \mathcal{J}(P_n)$. Note that

$$\delta_\alpha = \cup\{\gamma \mid \gamma \subset P_n(F), \alpha \cup \gamma \in \mathcal{J}(P_n)\}.$$

If we set $\beta = \alpha \cup \delta_\alpha$, then, by the definition of δ_α, β has the following property: for any $j \in F$, $p_j \notin \beta$ implies $\beta \cup \{p_j\} \notin \mathcal{J}(P_n)$.

Lemma 3.3. *Let $\emptyset \neq F \subsetneq [n]$ and let $\mathcal{S}$ be the set of poset ideals β of P_n with the property that for any $j \in F$ such that $p_j \notin \beta$ we have $\beta \cup \{p_j\} \notin \mathcal{J}(P_n)$. Then the map $\varphi\colon \mathcal{J}(P_n(\bar{F})) \to \mathcal{S}$ defined by $\alpha \mapsto \beta := \alpha \cup \delta_\alpha$, is an isomorphism of posets.*

Proof. φ is invertible. Indeed, the map $\psi : \mathcal{S} \to \mathcal{J}(P_n(\bar{F}))$ defined by $\psi(\beta) = \beta \cap P_n(\bar{F})$ is the inverse of φ since if $\alpha = \beta \cap P_n(\bar{F})$, then, by the property of β, we have $\delta_\alpha = \beta \setminus P_n(\bar{F})$.

Let $\alpha_1 \subsetneq \alpha_2$ be poset ideals of $P_n(\bar{F})$ and $\beta_i = \varphi(\alpha_i) = \alpha_i \cup \delta_i,\ i = 1, 2$. We only need to show that $\beta_1 \subset \beta_2$ since the strict inclusion follows from the hypothesis $\alpha_1 \subsetneq \alpha_2$. Let us assume that $\beta_1 \not\subset \beta_2$ and let $p_a, a \in F$, be a minimal element in $\beta_1 \setminus \beta_2$. Since $p_a \notin \beta_2$, it follows that $\beta_2 \cup \{p_2\}$ is not a poset ideal of P_n. Therefore there exists $p_b < p_a$ such that $p_b \notin \beta_2$. On the other hand, $p_b \in \beta_1$ since $\beta_1 \in \mathcal{J}(P_n)$, hence, $p_b \in \beta_1 \setminus \beta_2$, which leads to a contradiction with the choice of p_a.

Now let $\beta_1 \subsetneq \beta_2$, $\beta_1, \beta_2 \in \mathcal{S}$, and assume that $\alpha_1 = \alpha_2$, where $\alpha_1 = \beta_1 \cap P_n(\bar{F})$, and $\alpha_2 = \beta_2 \cap P_n(\bar{F})$. Then $\delta_1 = \beta_1 \setminus P_n(\bar{F}) \subsetneq \delta_2 = \beta_2 \setminus P_n(\bar{F})$. But this is impossible since δ_1 is maximal among the subsets γ of $P_n(F)$ such that $\alpha_1 \cup \gamma \in \mathcal{J}(P_n)$. $\square$

We can state now the main theorem which relates the Hilbert series of the vertex cover algebra $A(G)$ to the Hilbert series of the basic cover algebras $\bar{A}(G_F)$ for all $F \subset [n]$.

Theorem 3.4. *For $F \subset [n]$ let $H_{\bar{A}(G_F)}(z)$ be the Hilbert series of $\bar{A}(G_F)$ and let $H_{A(G)}(z)$ be the Hilbert series of $A(G)$. Then:*

$$H_{A(G)}(z) = \frac{1}{(1-z)^n} \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z}\right)^{n-|F|}. \tag{6}$$

In particular, if $h(z) = \sum_{j\geq 0} h_j z^j$ and $h^F(z) = \sum_{j\geq 0} h_j^F z^j$, where $h = (h_j)_{j\geq 0}$ and $h^F = (h_j^F)_{j\geq 0}$ are the h-vectors of $A(G)$, and, respectively, $\bar{A}(G_F)$, then

$$h(z) = \sum_{F \subset [n]} h^F(z) z^{n-|F|}. \tag{7}$$

Proof. Let $J_G = \mathrm{in}_{<^{\#}_{lex}}(Q_G)$. It is known that B_G/Q_G and B_G/J_G have the same Hilbert series. Let $B'_G = K[\{x_i\}_{1\leq i\leq n}, \{u_\alpha\}_{\alpha\in\mathcal{J}(P_n)}]$. By using the following K-vector space isomorphism

$$B_G/J_G \simeq K[y_1, y_2, ..., y_n] \otimes_K B'_G/(J_G \cap B'_G),$$

we get

$$H_{A(G)}(z) = H_{B_G/Q_G}(z) = H_{B_G/J_G}(z) = \frac{1}{(1-z)^n} H_{B'_G/(J_G\cap B'_G)}(z).$$

We need to compute the Hilbert series of $B'_G/(J_G \cap B'_G)$. To this aim we show that we have an isomorphism of K-vector spaces

$$B'_G/(J_G \cap B'_G) \simeq \bigoplus_{F\subset[n]} \bar{A}(G_{\bar{F}}) \otimes_K x_F K[\{x_i\}_{i\in F}]. \tag{8}$$

For $\emptyset \neq F \subset [n]$ let J_F be the initial ideal with respect to $<^{\#}$ of the toric ideal of $\bar{A}(G_F)$. Then $J_F = (u_\alpha u_\beta | \alpha, \beta \in \mathcal{J}(P_n(F)), \alpha \not\subset \beta, \beta \not\subset \alpha)$. If $F = \emptyset$, we put by convention $J_F = (u_\emptyset)$.

The basic vertex cover algebra $\bar{A}(G_{\bar{F}})$ can be decomposed as a K-vector space as $\bar{A}(G_{\bar{F}}) \simeq \bigoplus_{w\notin J_{\bar{F}}} Kw$. We notice that $w \notin J_{\bar{F}}$ if and only if $\mathrm{supp}(w) = \{\alpha_1, ..., \alpha_s\}$,

$s \geq 0$, where $\alpha_1 \subsetneq \ldots \subsetneq \alpha_s$ is a chain in $\mathcal{J}(P_n(\bar{F}))$. It follows that for $F \subset [n]$ we have

$$V_F := \bar{A}(G_{\bar{F}}) \otimes_K x_F K[\{x_i\}_{i \in F}] \simeq \bigoplus K v w,$$

where the direct sum is taken over all monomials vw with v monomial in the variables x_i such that $\mathrm{supp}(v) = F$ and w monomial in the variables u_α such that $w \notin J_{\bar{F}}$. As a K-vector space, $B'_G/(J_G \cap B'_G)$ has the decomposition

$$B'_G/(J_G \cap B'_G) \simeq \bigoplus_{F \subset [n]} \bigoplus W_F,$$

where $W_F = \bigoplus K v w'$ and the direct sum is taken over all monomials v with $\mathrm{supp}(v) = F$ and all monomials w' in the variables u_α with $\alpha \in \mathcal{J}(P_n)$ such that $vw' \neq 0$ modulo $J_G \cap B'_G$.

In order to prove (8), we only need to show that for each $F \subset [n]$, the K-vector spaces V_F and W_F are isomorphic. This is obvious for $F = \emptyset$ and $F = [n]$.

Let us consider now $\emptyset \neq F \subsetneq [n]$. Based on the previous lemma, we are going to show that there exists a bijection between the K-bases of V_F and W_F.

Let vw be an element of the K-basis of V_F. This means that $\mathrm{supp}(v) = F$ and w is of the form $w = u_{\alpha_1}^{a_1} \cdots u_{\alpha_s}^{a_s}$ for some chain $\alpha_1 \subsetneq \ldots \subsetneq \alpha_s$ in $\mathcal{J}(P_n)$, $s \geq 1$. For each $1 \leq i \leq s$, let $\beta_i = \varphi(\alpha_i) \in \mathcal{J}(P_n)$ as it was defined in Lemma 3.3. We map vw to the monomial vw' where $w' = u_{\beta_1}^{a_1} \cdots u_{\beta_s}^{a_s}$. By Lemma 3.3, we have that $\beta_1 \subsetneq \ldots \subsetneq \beta_s$ is a chain in $\mathcal{J}(P_n)$. Moreover, for any $j \in F$ and any β_i such that $p_j \notin \beta_i$, we have $\beta_i \cup \{p_j\} \notin \mathcal{J}(P_n)$. Therefore, vw' is a monomial in the K-basis of W_F.

Conversely, let vw' be a monomial from the K-basis of W_F, where $\mathrm{supp}(v) = F$ and $w' = u_{\beta_1}^{a_1} \cdots u_{\beta_s}^{a_s}$, with $\beta_1 \subsetneq \ldots \subsetneq \beta_s$ a chain in $\mathcal{J}(P_n)$. Let $\alpha_i = \beta_i \cap P_n(\bar{F})$, for $1 \leq i \leq s$. Then we associate to vw' the monomial vw in the K-basis of V_F, where $w = u_{\alpha_1}^{a_1} \cdots u_{\alpha_s}^{a_s}$.

By using again Lemma 3.3 it follows that the above defined maps between the K-bases of V_F and W_F are inverse.

By (8) we get

$$H_{B'_G/(J_G \cap B'_G)}(z) = \sum_{F \subset [n]} H_{\bar{A}(G_{\bar{F}})}(z) \left(\frac{z}{1-z}\right)^{|F|} = \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z}\right)^{n-|F|}.$$

Hence

$$H_{A(G)}(z) = \frac{1}{(1-z)^n} \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z}\right)^{n-|F|}.$$

Since $H_{A(G)}(z) = \frac{h(z)}{(1-z)^{2n+1}}$ and $H_{\bar{A}(G_F)} = \frac{h^F(z)}{(1-z)^{n+1}}$, for all $F \subset [n]$, it follows that $h(z) = \sum\limits_{F \subset [n]} h^F(z) z^{n-|F|}$. □

Corollary 3.5. *For all $0 \leq j \leq n-1$, the j-th component h_j of the h-vector of $A(G)$ is equal to the number of all linear extensions of all $n-l$-element subposets of P_n, which, seen as permutations of $[n-l]$, have exactly $j-l$ descents, for all $0 \leq l \leq j$.*

Proof. It follows immediately from (7) and Remark 3.2. $\square$

Corollary 3.6. *The h-vector of $A(G)$ is unimodal.*

Proof. By (7) we get $h_{n+1} = \sum_{F \subset [n]} h^F_{|F|+1}$ and $h_n = \sum_{F \subset [n]} h^F_{|F|}$. By using (5) from Remark 3.2, we have $h^F_{|F|} = h^F_{|F|+1} = 0$, for all $\emptyset \neq F \subset [n]$. Hence $h_{n+1} = h^{\emptyset}_1 = 0$ and $h_n = h^{\emptyset}_0 = 1$. In [5, Corollary 4.4] it is proved that $A(G)$ is a Gorenstein ring, hence, by [2, Corollary 4.3.8 (b) and Remark 4.3.9 (a)], $h_i = h_{n-i}$, for all $0 \leq i \leq n$. We denote by $\nu(l,j)$ the number of all linear extensions of all $n-l$-element subposets of P_n which, seen as permutations of $[n-l]$, have exactly $j-l$ descents. Hence, by Corollary 3.5, $h_j = \sum_{l=0}^{j} \nu(l,j)$. Let $0 \leq j < j+1 \leq \lfloor \frac{n}{2} \rfloor$. Then $\nu(l,j) \leq \nu(l+1,j+1)$, for all $0 \leq l \leq j$, which implies that $h_{j+1} = \nu(j+1,0) + \sum_{l=0}^{j} \nu(j+1,l+1) \geq \sum_{l=0}^{j} \nu(l,j) = h_j$. $\square$

Remark 3.7. The Hilbert series of the vertex cover algebra $A(G)$ is given by

$$H_{A(G)}(z) = \frac{h_0 + h_1 z + \ldots + h_{n-1} z^{n-1} + h_n z^n}{(1-z)^{2n+1}},$$

where $h = (h_0, \ldots, h_n)$ is the h-vector of $A(G)$. In particular, we recover the known fact that $\dim A(G) = 2n+1$. It also follows that the a-invariant is $a = -n-1$.

Corollary 3.8. *Let $e(A(G))$ be the multiplicity of $A(G)$ and let $e(\bar{A}(G_F))$ the multiplicity of $\bar{A}(G_F)$ for $F \subset [n]$. Then*

$$e(A(G)) = \sum_{F \subset [n]} e(\bar{A}(G_F)).$$

Proof. It follows immediately from (7). $\square$

Let $P_3 = \{p_1, p_2, p_3\}$ be the poset with $p_1 \leq p_2$ and $p_1 \leq p_3$ and $G_3 = G(P_3)$. Then $H_{\bar{A}(G_{\emptyset})}(z) = \frac{1}{1-z}$, $H_{\bar{A}(G_{\{1\}})}(z) = H_{\bar{A}(G_{\{2\}})}(z) = H_{\bar{A}(G_{\{3\}})}(z) = \frac{1}{(1-z)^2}$, $H_{\bar{A}(G_{\{1,2\}})}(z) = H_{\bar{A}(G_{\{1,3\}})}(z) = \frac{1}{(1-z)^3}$, $H_{\bar{A}(G_{\{2,3\}})}(z) = \frac{1+z}{(1-z)^3}$, $H_{\bar{A}(G_{\{1,2,3\}})}(z) = \frac{1+z}{(1-z)^4}$ and the Hilbert series of $A(G_3)$ is:

$$H_{A(G_3)}(z) = \frac{1}{(1-z)^3} \sum_{F \subset [3]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z} \right)^{3-|F|} = \frac{z^3 + 4z^2 + 4z + 1}{(1-z)^7}.$$

Hence $h_0 = h_3 = 1$, $h_1 = h_2 = 4$, $h_4 = 0$, $e(A(G_3)) = 10$. We can also compute the h-vector of $A(G_3)$ by using Corollary 3.5. The poset P_3 has two linear extensions, which, seen as permutation of [3], are equal to id_3 and (23). Hence $h_0 = 1$, since there exists only one linear extension of P_3, which, seen as a permutation of [3], has exactly 0 descents. Furthermore, P_3 has three 2-element subposets, the chains $P_3(\{1,2\})$ and $P_3(\{1,3\})$ with a linear extension corresponding to id_2, and the antichain $P_3(\{2,3\})$ with two linear extensions corresponding to id_2 and (12). Thus $h_1 = 4$, since there exists only one linear extension of P_3, which, seen as a permutation of [3], has exactly 1 descent and each of the subposets $P_3(\{1,2\})$, $P_3(\{1,3\})$

and $P_3(\{2,3\})$ has one linear extension, which, seen as a permutation of $[2]$, has exactly 0 descents.

Let $\mathcal{L}_n$ be the Boolean lattice on $\{p_1, p_2, ..., p_n\}$, $n \geq 1$, and $A(p,q)$ be the Eulerian number for $1 \leq q \leq n$ and $0 \leq p < q$. By convention, we put $A(0,0) = 1$ and $A(q,q) = 0$, for all $1 \leq q \leq n$.

We compute the Hilbert series of the vertex cover algebra of the Cohen-Macaulay bipartite graphs that come from a chain and an antichain.

Proposition 3.9. *Let G' be a bipartite graph that comes from a chain and G'' a bipartite graph that comes from an antichain with n elements, $n \geq 1$. Then we have*

(i) $H_{A(G')}(z) = \frac{(1+z)^n}{(1-z)^{2n+1}}$. *In particular,* $e(A(G')) = 2^n$.

(ii) $H_{A(G'')}(z) = \frac{\sum_{j=0}^{n}\sum_{l=0}^{j}\binom{n}{l}A(n-l,j-l)z^j}{(1-z)^{2n+1}}$. *In particular,* $e(A(G'')) = n! \cdot \sum_{l=0}^{n} \frac{1}{l!}$.

Proof. (i) We may assume that $G' = G(P'_n)$, where $P'_n = \{p'_1, p'_2, ..., p'_n\}$ is the chain with $p'_1 \leq p'_2 \leq ... \leq p'_n$. P'_n as well as all its subposets have a unique linear extension. Therefore, the h-vector of G' is $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

(ii) Let $G'' = G(P''_n)$, where $P''_n = \{p''_1, ..., p''_n\}$ is an antichain. If $F = [n]$, then, by convention, $A(0,0) = 1 = h_0^{\bar{F}}$. If $F \subsetneqq [n]$, then $\mathcal{J}(P''_n(\bar{F}))$ is a Boolean lattice on the set $P''_n(\bar{F})$, which implies that $\mathcal{J}(P''_n(\bar{F}))$ is isomorphic to $\mathcal{L}_{n-l}$, where $l = |F|$. Therefore, by Remark 3.2, $h_i^{\bar{F}} = A(n-l, i)$, for all $0 \leq i \leq n-l-1$. If $i = n-l$, then $A(n-l, i) = 0$ (by convention) and $h_i^{\bar{F}} = 0$ (by Remark 3.2), which implies that $A(n-l,i) = h_i^{\bar{F}}$. By (6) we have $h''_j = \sum_{l=0}^{j} \sum_{\substack{F \subset [n] \\ |F| = l}} h_{j-l}^{\bar{F}}$, hence $h''_j = \sum_{l=0}^{j} \binom{n}{l} A(n-l, j-l)$, for all $0 \leq j \leq n$.

We get $e(A(G'')) = \sum_{j=0}^{n} h''_j = \sum_{j=0}^{n-1} h''_j + 1 = \sum_{j=0}^{n-1}\sum_{l=0}^{j} \binom{n}{l} A(n-l, j-l) + 1 = \sum_{l=0}^{n-1} \binom{n}{l} \cdot \sum_{j=0}^{n-l-1} A(n-l, j) + 1$. We obviously have $\sum_{j=0}^{n-l-1} A(n-l, j) = (n-l)!$, for all $0 \leq l \leq n-1$. Therefore, $e(A(G'')) = \sum_{l=0}^{n-1} \binom{n}{l} \cdot (n-l)! + 1 = n! \cdot \sum_{l=0}^{n} \frac{1}{l!}$. $\square$

Remark 3.10. The reduced Gröbner basis $\mathcal{G}'$ of the toric ideal $Q_{G'}$ of $A(G')$ with respect to the monomial order $<^{\sharp}_{lex}$ on the polynomial ring $B_{G'}$ is:

$$\mathcal{G}' = \{\underline{x_j u_{\{p'_1, ..., p'_{j-1}\}}} - y_j u_{\{p'_1, ..., p'_j\}} | j \in [n]\},$$

where the initial monomial of each binomial of $\mathcal{G}'$ is the first monomial.

We notice that the initial ideal $\mathrm{in}_{<^{\sharp}_{lex}}(Q_{G'}) = (x_j u_{\{p'_1, ..., p'_{j-1}\}} | j \in [n])$ is a complete intersection, which implies that the toric ideal $Q_{G'}$ is a complete intersection. Thus $A(G')$ has a pure resolution given by the Koszul complex.

Proposition 3.11. *Let G be a Cohen-Macaulay bipartite graph on V_n, $n \geq 1$. Then the following assertions hold:*

(i) *G comes from a chain if and only if* $H_{A(G)}(z) = \frac{(1+z)^n}{(1-z)^{2n+1}}$;

(ii) *G comes from an antichain if and only if* $H_{A(G)}(z) = \frac{h''_n z^n + h''_{n-1} z^{n-1} ... + h''_1 z + h''_0}{(1-z)^{2n+1}}$, *where $h'' = (h''_0, h''_1, \ldots, h''_n)$ is the h-vector of the vertex cover algebra $A(G'')$ of the bipartite graph G'' that comes from an antichain $P''_n = \{p''_1, p''_2, \ldots, p''_n\}$.*

Proof. Let us suppose that G comes from a poset $P_n = \{p_1, p_2, ..., p_n\}$, $n \geq 1$, and let $h = (h_0, h_1, ..., h_n)$ be the h-vector of $A(G)$. In the first place we need to compute the component h_1. By using (7), we get $h_1 = h_1^{[n]} + n$. But $h_1^{[n]}$ is the component of rank 1 in the h-vector of $\bar{A}(G)$. By using the formula which relates the h-vector to the f-vector for the order complex $\Delta(\mathcal{J}(P_n))$, we immediately get $h_1^{[n]} = |\mathcal{J}(P_n)| - n - 1$, which implies that $h_1 = |\mathcal{J}(P_n)| - 1$.
(i) Let $h_1 = n$. Then $|\mathcal{J}(P_n)| = n + 1$, which implies that P_n is a chain.
(ii) Let $h_1 = h''_1 = |\mathcal{J}(P''_n)| - 1 = 2^n - 1$. Then $|\mathcal{J}(P_n)| = 2^n$, which implies that P_n is an antichain.

In both cases the converse follows from Proposition 3.9. □

Proposition 3.12. *Let G be a Cohen-Macaulay bipartite graph on V_n, $n \geq 1$. If $h = (h_0, h_1, ..., h_n)$ is the h-vector of $A(G)$, then $\binom{n}{j} \leq h_j \leq h''_j$, for all $0 \leq j \leq n$, where G'' comes from an antichain with n elements and $h'' = (h''_0, h''_1, ..., h''_n)$ is the h-vector of $A(G'')$.*

Proof. We may assume without loss of generality that $G = G(P_n)$, where $P_n = \{p_1, ..., p_n\}$ is a poset such that $p_i \leq p_j$ implies $i \leq j$. Let $P'_n = \{p'_1, p'_2, ..., p'_n\}$ the chain with $p'_1 \leq p'_2 \leq ... \leq p'_n$ and $P''_n = \{p''_1, p''_2, ..., p''_n\}$ an antichain. By using (7) and (5), we get $h_0 = 1 = h''_0$ and $h_n = 1 = h''_n$. Let $1 \leq j \leq n-1$. By Corollary 3.5, h_j is equal to the number of all linear extensions of all $n-l$-element subposets, which, seen as permutations of $[n-l]$, have exactly $j - l$ descents, for all $0 \leq l \leq j$. Each $n - l$-element subposet of P'_n, respectively, P''_n is a chain, respectively, an antichain, hence it has only one linear extension which corresponds to id_{n-l}, respectively, it has $(n-l)!$ linear extensions which correspond to all permutations of $[n-l]$. Therefore $\binom{n}{j} \leq h_j \leq h''_j$, for all $1 \leq j \leq n-1$. □

Corollary 3.13. *Let G be a bipartite graph that comes from a poset with n elements, $n \geq 1$. Then $2^n \leq e(A(G)) \leq n! \sum\limits_{l=0}^{n} \frac{1}{l!}$. The left equality holds if and only if the poset is a chain and the right equality holds if and only if the poset is an antichain.*

Proof. Let $G' = G(P'_n)$ and $G'' = G(P''_n)$, where $P'_n = \{p'_1, p'_2, \ldots, p'_n\}$ is a chain and $P''_n = \{p''_1, p''_2, \ldots, p''_n\}$ is an antichain. We may assume without loss of generality that $p'_1 \leq p'_2 \leq ... \leq p'_n$ and $G = G(P_n)$, where $P_n = \{p_1, p_2, ..., p_n\}$ is a poset such that $p_i \leq p_j$ implies $i \leq j$. Let h, h', respectively, h'' be the h-vector of $A(G)$, $A(G')$, respectively, $A(G'')$. By summing up the inequalities $h'_j \leq h_j \leq h''_j$ from Proposition 3.12 or by applying Corollary 2.6, we obtain $e(A(G')) \leq e(A(G)) \leq e(A(G''))$. Next, from Proposition 3.9, we get the desired inequalities. The left equality, respectively, the right equality holds if and only if $h'_j = h_j$, respectively, $h_j = h''_j$, for all $0 \leq j \leq n$, therefore, by using Proposition 3.11, this is equivalent to $P_n = P'_n$, respectively, $P_n = P''_n$. □

Acknowledgment

I would like to thank Professor Jürgen Herzog for very useful suggestions and discussions on the subject of this paper. I am also very grateful to Professor Volkmar Welker who explained to me the combinatorial significance of the h-vector of a Hibi ring.

References

[1] B. Benedetti, A. Constantinescu, M. Varbaro - *Dimension, depth and zero-divisors of the algebra of basic k-covers of a graph*, Preprint, 2009, arXiv.org: 0901.3895;

[2] W. Bruns, J. Herzog - *Cohen-Macaulay Rings*, rev. ed., Cambridge Stud. Adv. Math., 39, Cambridge Univ. Press, Cambridge, 1998;

[3] J. Herzog, T. Hibi - *Distributive lattices, Bipartite Graphs and Alexander Duality*, J. Algebraic Combin. **22** (2005), 289–302;

[4] J. Herzog, T. Hibi, H. Ohsugi - *Unmixed bipartite graphs and sublattices of the boolean lattices*, Preprint, 2008, arXiv.org: 0806.1088;

[5] J. Herzog, T. Hibi, N. V. Trung - *Symbolic powers of monomial ideals and vertex cover algebras*, Adv. Math. **210** (2007), 304–322;

[6] V. Reiner, V. Welker - *On the Charney-Davis and Neggers-Stanley conjectures*, J. Combin. Theory, **109** (2005), 247-280;

[7] A. Van Tuyl, R.H. Villareal - *Shellable graphs and sequentially Cohen-Macaulay bipartite graphs*, J. Combin. Theory Ser. A **115** (2008), 799-814;

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